Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet answers for 2021-11-08

If you would like clarification on any problems, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1.

(a) This one definitely makes sense—you've used it a lot! The output is a vector field.
(b) div **F** makes sense and outputs a scalar function. But you can't take the curl of that, so the entire expression is nonsense.
(c) This expression makes sense and outputs a vector field.
(d) $\nabla \times f$ doesn't make sense; you can't take the curl of a scalar function.
(e) This makes sense and outputs a scalar function. In fact this is called the Laplacian, sometimes denoted $\nabla^2 f$.
(f) This makes sense and outputs a scalar function.
(g) This makes sense and outputs a scalar function.

Question 2.

- (a) makes sense and certainly can be nonzero, e.g. take $f(x, y, z) = x$.
- (c) makes sense but is always zero because $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$.
- (e) makes sense and can be nonzero, e.g. take $f(x, y, z) = x^2$.
- (f) makes sense and is always zero since $\nabla \cdot (\nabla \times (\text{vector field})) = 0$.
- (g) makes sense and is always zero too; using the provided identity you can rewrite it as

$$\nabla g \cdot (\nabla \times \nabla f) - \nabla f \cdot (\nabla \times \nabla g)$$

but both $\nabla \times \nabla f$ and $\nabla \times \nabla g$ are equal to $\mathbf{0}$ (the zero vector field).

Answers to computations

Problem 1. We proceed using the identities provided.

(a)

$$\begin{aligned}\nabla(r^n) &= \nabla((r^2)^{n/2}) \\ &= \frac{n}{2}(r^2)^{\frac{n}{2}-1}2\mathbf{r} \\ &= \boxed{nr^{n-2}\mathbf{r}}.\end{aligned}$$

(b)

$$\begin{aligned}\nabla \times (r^{n-1}\mathbf{r}) &= (\nabla r^{n-1}) \times \mathbf{r} + r^{n-1}(\nabla \times \mathbf{r}) \\ &= (n-1)r^{n-3}\mathbf{r} \times \mathbf{r} + \mathbf{0} \\ &= \boxed{\mathbf{0}}.\end{aligned}$$

where we have used that $\mathbf{r} \times \mathbf{r} = \mathbf{0}$ (because the cross product of parallel vectors is zero), and also that $\nabla \times \mathbf{r} = \mathbf{0}$ (easily checked by direct computation).

Alternatively, if $n \neq -1$, we could note that

$$r^{n-1}\mathbf{r} = \nabla(\frac{1}{n+1}r^{n+1})$$

using part (a). This means that $r^{n-1}\mathbf{r}$ is conservative, so it must have zero curl:

$$\nabla \times (r^{n-1}\mathbf{r}) = \nabla \times \nabla(\frac{1}{n+1}r^{n+1}) = \mathbf{0}.$$

(It's still conservative when $n = -1$, but in that case the potential function will involve ln.)

(c)

$$\begin{aligned}\nabla \cdot (r^{n-1}\mathbf{r}) &= (\nabla r^{n-1}) \cdot \mathbf{r} + r^{n-1}(\nabla \cdot \mathbf{r}) \\ &= (n-1)r^{n-3}\mathbf{r} \cdot \mathbf{r} + 3r^{n-1} \\ &= (n-1)r^{n-1} + 3r^{n-1} = \boxed{(n+2)r^{n-1}}.\end{aligned}$$

This scalar function is identically zero if and only if $n = -2$. Incidentally this is the exponent that appears in physics: many vector fields have a fall-off which is proportional to $(\text{distance})^{-2}$.